Stationarity and Reversibility

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1 Why are Definitions Important?

In section, we noted that the motivation for definitions like *irreducibility*, *aperiodicity*, and *recurrence* was to ensure that a stationary distribution existed and was unique. Moreover, given the "stationary" title, we'd want the Markov chain to converge to the distribution in the "long-run", or as $n \to \infty$. We will explore why these definitions are necessary through two simple examples.

Proposition 1.1 *Any finite-state, homogeneous Markov chain has a stationary distribution.*

While the proof is a little technically involved for the purposes of this course, it's in fact true that for all Markov chains we'll be concerned with (i.e. finite state space, time-homogeneous so transition matrix exists), a stationary distribution exists. However, it may not be unique.

Example. Consider the simple Markov chain given by the transition matrix:

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In other words, it stays in its current state with probability 1. Note that *any distribution* over these two states is a stationary distribution under this transition matrix! So the stationary distribution is clearly not unique.

So we would like to examine when a stationary distribution is in fact unique. It turns out that there is a simple condition.

Proposition 1.2 *Any finite-state, homogeneous Markov chain that is irreducible has a unique stationary distribution,*

Note that in the example above, you couldn't reach either state from the other, so it was in fact reducible. As long as we can reach any state from any other state (i.e. *irreducible*), that is, all states are recurrent with each other, the Markov chain has a *unique* stationary distribution.

However, this doesn't necessarily mean that the distribution over the states, given some arbitrary initial state, will *converge to* the stationary distribution in the long-run!

Example. Consider another simple Markov chain with the transition matrix:

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This Markov chain switches states at every time point with probability 1. Now consider initial state $X_0 = 1$. Note that the distribution over the states is:

$$X_n = \begin{cases} 1 & \text{if n is even} \\ 2 & \text{if n is odd} \end{cases}$$

Thus, the distribution emphatically does not converge to any stationary distribution! One additional condition ensures that this does not happen.

Proposition 1.3 *Any finite-state, homogeneous Markov chain that is irreducible and aperiodic has a unique stationary distribution* π*, and* $\lim_{n\to\infty} P(X_n) = \pi$.

In other words, adding the condition of *aperiodicity* ensures that the long-run distribution over the states converges to the stationary distribution, regardless of the initial state/distribution. Recall that a Markov chain is *aperiodic* if and only if *every state* is aperiodic.

2 Reversibility

Solving the equations $\pi = \pi Q$ can be troublesome when the state space is large. Another definition, *reversibility*, helps us in these times of need.

Definition A Markov chain with transition matrix Q is *reversible* with respect to a probability distribution over the states π if $\pi_i Q_{ij} = \pi_j Q_{ji}$.

Proposition 2.1 *If a Markov chain is reversible with respect to distribution* π*, then* π *is a stationary distribution for the Markov chain.*

Proof I provide a quick proof of this because I didn't quite know why this was true for a while, but it turns out that it's rather simple. Recall that $\sum_k Q_{jk} = 1$, because all states must transition somewhere. Thus, $Q_{ji} = 1 - \sum_{k\neq i} Q_{jk}$, and so:

$$\pi_j Q_{ji} = \pi_j \left(1 - \sum_{k\neq i} Q_{jk}\right) = \pi_j - \sum_{k\neq i} \pi_j Q_{jk} = \pi_j - \sum_{k\neq i} \pi_k Q_{kj}$$

where we applied the reversibility condition to each of the terms in the sum in the last equality. However, inputting this into our original reversibility condition of $\pi_i Q_{ij} = \pi_j Q_{ji}$, we have:

$$\pi_i Q_{ij} = \pi_j - \sum_{k\neq i} \pi_k Q_{kj}$$

$$\Rightarrow \pi_i Q_{ij} + \sum_{k\neq i} \pi_k Q_{kj} = \pi_j$$

$$\Rightarrow \pi_j = \sum_k \pi_k Q_{kj}$$

which is exactly the condition needed for stationarity! ∎